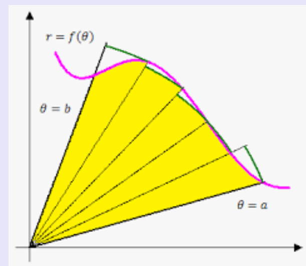
Calculus II

Lecture 24



Feb 19-8:47 AM

Class QZ 22

Determine whether the Series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ Converges or diverges.

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

Diverges by P-Series $P=1$

$$\frac{1}{\sqrt{n^2+1}} < \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

$$a_n < b_n$$

limit Comparison Test $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

both diverge $\rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ Diverges

Jul 23-6:48 AM

Suppose $f(x)$ can be written as a Power Series around a such that $|x-a|<R$

$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + \cdots$

$f(a) = C_0 + C_1(a-a) + C_2(a-a)^2 + C_3(a-a)^3 + \cdots$

$\boxed{C_0 = f(a)}$

$f'(x) = C_1 + 2C_2(x-a)^1 + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \cdots$

$f'(a) = C_1 + 2C_2(a-a) + 3C_3(a-a)^2 + 4C_4(a-a)^3 + \cdots$

$\boxed{C_1 = f'(a)}$ $\quad C_1 = \dfrac{f'(a)}{1} \rightarrow 1!$

$f''(x) = 2C_2 + 3\cdot 2C_3(x-a)^1 + 4\cdot 3C_4(x-a)^2 + \cdots$

$f''(a) = 2C_2 \quad C_2 = \dfrac{f''(a)}{2} \rightarrow 2!$

$f'''(x) = 3\cdot 2C_3 + 4\cdot 3\cdot 2C_4(x-a)^1 + \cdots$

$f'''(a) = 3\cdot 2C_3 \quad C_3 = \dfrac{f'''(a)}{3\cdot 2} \rightarrow 3!$

$f^{(4)}(a) = 4\cdot 3\cdot 2C_4 \quad C_4 = \dfrac{f^{(4)}(a)}{4\cdot 3\cdot 2} \rightarrow 4!$

$f^{(5)}(a) = 5\cdot 4\cdot 3\cdot 2C_5 \quad C_5 = \dfrac{f^{(5)}(a)}{5\cdot 4\cdot 3\cdot 2} \rightarrow 5!$

Jul 23-8:17 AM

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad C_n = \frac{f^{(n)}(a)}{n!}}$$

$C_6 = \dfrac{f^{(6)}(a)}{6!}$

Taylor's Series around $x=a$

$C_0 = \dfrac{f^{(0)}(a)}{0!} = \dfrac{f(a)}{1}$

If $a=0$ $\rightarrow$ $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Maclaurin Series

Jul 23-8:28 AM

$f(x) = e^x \qquad a = 0 \qquad e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$

$f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = 1 \qquad = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$f''(x) = e^x \qquad f''(0) = 1$

$f'''(x) = e^x \qquad f'''(0) = 1 \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Find Interval of Convergence

Use Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}\cdot n!}{x^n\cdot(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = |x|\cdot\underbrace{\lim_{n\to\infty}\frac{1}{n+1}}_{\to 0}$$

Converges $(-\infty,\infty)$

can be any R.

$= |x|\cdot 0 = \boxed{0}$

Convergence

Jul 23-8:33 AM

Find power series xe^{x^2}.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty}\frac{(x^2)^n}{n!}, \quad xe^{x^2} = x\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!}}$$

Jul 23-8:42 AM

Find a power Series for $f(x) = \ln(1+x)$

1) From Yesterday

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$
$$= \int \boxed{\frac{1}{1-(-x)}}\,dx$$
$$= \int \left[1 - x + x^2 - x^3 + x^4 - \cdots\right] dx$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots + C$$

for $x=0$ $\quad \cancel{\ln 1} \to 0 = 0 - 0 + 0 - 0 + \cdots + C$

$\boxed{C=0}$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Jul 23-8:44 AM

2) Now today near $0 \to a=0$

$f(x) = \ln(1+x) \qquad f(0) = \ln 1 = 0$

$f'(x) = \dfrac{1}{1+x} = (1+x)^{-1} \qquad f'(0) = \dfrac{1}{1+0} = 1 = 0!$

$f''(x) = \dfrac{-1}{(1+x)^2} = -1\cdot(1+x)^{-2} \qquad f''(0) = \dfrac{-1}{(1+0)^2} = -1 = -1!$

$f'''(x) = 2(1+x)^{-3} \qquad f'''(0) = 2 = 2!$

$f^{(4)}(x) = -3\cdot 2(1+x)^{-4} \qquad f^{(4)}(0) = -3\cdot 2 = 3!$

$$\ln(1+x) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \cdots$$

$$\ln(1+x) = 0 + \frac{1}{1}\cdot x + \frac{-1!}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{-3!}{4!}x^4$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Jul 23-8:49 AM

Find Taylor Series for $f(x)=\sqrt{x}$ near $\boxed{x=4}$

$f(x)=\sqrt{x}$ $\quad a=4$

$f(a)=\sqrt{4}=2$

$f'(x)=\frac{1}{2\sqrt{x}} \Rightarrow f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{2\cdot 2}=\frac{1}{4}$ $\quad \frac{1}{2}x^{-1/2}$

$f''(x)=\frac{1}{2}\cdot\frac{-1}{2}\cdot x^{-3/2}=\frac{-1}{4(x\sqrt{x})}$ $\quad x^{-3/2}$ $\quad f''(4)=\frac{-1}{32}$

$f'''(x)=\frac{-1}{4}\cdot\frac{-3}{2}\cdot x^{-5/2}=\frac{3}{8x^2\sqrt{x}}$ $\quad f'''(4)=\frac{3}{256}$

$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3$

$\sqrt{x}=2+\frac{1}{4}(x-4)+\frac{\frac{-1}{32}}{2}(x-4)^2+\frac{\frac{3}{256}}{6}(x-4)^3+\cdots+\cdots$

$=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3+\cdots$

Evaluate

$\sqrt{4.1}\approx 2+\frac{1}{4}(\overset{.1}{4.1-4})-\frac{1}{64}(\overset{.1}{4.1-4})^2+\frac{1}{512}(\overset{.1}{4.1-4})^3$

$=2+\frac{1}{40}-\frac{1}{6400}+\frac{1}{512000}$

From Calc $2.025 \approx 2.025$

Jul 23-9:00 AM

Find a power series for $f(x)=\frac{1}{(1-x)^2}$

Relationship between $\frac{1}{(1-x)^2}$ & $\frac{1}{1-x}$

$\frac{d}{dx}\left[\frac{1}{1-x}\right]=\frac{0\cdot(1-x)-1(-1)}{(1-x)^2}=\frac{1}{(1-x)^2}$

$\frac{d}{dx}\left[1+x+x^2+x^3+\cdots+\cdots\right]=\frac{1}{(1-x)^2}$

$1+2x+3x^2+4x^3+\cdots=\frac{1}{(1-x)^2}$

$$\sum_{n=0}^{\infty}(n+1)x^n=\frac{1}{(1-x)^2}$$

Jul 23-9:13 AM

Find Maclaurin Series for $f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$

Taylor Series with $a=0$

$f(0) = \frac{1}{(1-0)^2} = \frac{1}{1} = 1 = 1!$

$f'(x) = -2(1-x)^{-3}\cdot(-1) = 2(1-x)^{-3}$ $\quad f'(0) = 2\cdot 1 = 2!$

$f''(x) = -3\cdot 2(1-x)^{-4}\cdot(-1) = 3\cdot 2(1-x)^{-4}$ $\quad f''(0) = 3\cdot 2 = 3!$

$f'''(x) = -4\cdot 3\cdot 2(1-x)^{-5}\cdot(-1) = 4\cdot 3\cdot 2\cdot(1-x)^{-5}$ $\quad f'''(0) = 4!$

Taylor Series

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + \frac{3!}{2!}x^2 + \frac{4!}{3!}x^3 + \frac{5!}{4!}x^4 + \cdots$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$= \sum_{n=0}^{\infty}(n+1)x^n$$

Jul 23-9:20 AM

Find a power Series for $f(x) = x\sin x$

work with $f(x) = \sin x$ Near 0.

$f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x$ $\quad f^{(5)}(0) = 1$

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= \cancel{0}^{\,0} + 1x + \cancel{\frac{0}{2!}x^2}^{\,0} + \frac{-1}{3!}x^3 + \cancel{\frac{0}{4!}x^4}^{\,0} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

$$x\sin x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{(2n-1)!}$$

Jul 23-9:30 AM

Find Maclaurin Series for $\cos x$.

$$\frac{d}{dx}\left[\sin x\right] = \cos x$$

$$\frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right] = \cos x$$

$$1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} + \ldots = \cos x$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots = \cos x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Jul 23-9:41 AM

Find Maclaurin Series for $f(x) = \tan^{-1} x$

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \left[1 - x^2 + x^4 - x^6 + x^8 - \ldots\right] dx$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \ldots + C$$

at $x=0$ $\quad \tan^{-1} 0 = 0 - 0 + 0 - 0 + 0 + \ldots C$

$$\boxed{0 = C}$$

$$\tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$$

from book

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Jul 23-9:51 AM

Class QZ 23

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges or diverges.

$a_n = \frac{1}{\sqrt{n}}$

1) $a_{n+1} < a_n$

2) $\lim_{n \to \infty} a_n = 0$

It converges by A.S.T.

Jul 23-9:58 AM